Analytical Mechanics

Review of angular momentum

Chpt. 3 in Kibble & Berkshire

$$\vec{J} \equiv \vec{r} \times \vec{p} = m\,\vec{r} \times \dot{\vec{r}}$$

Ang. mom. <u>about the origin</u> of a particle at position $\vec{r}$, and moving with momentum $\vec{p}$. $\vec{J}$ depends on ~~axis~~ point about which we evaluate it

Rate of change of $\vec{J}$

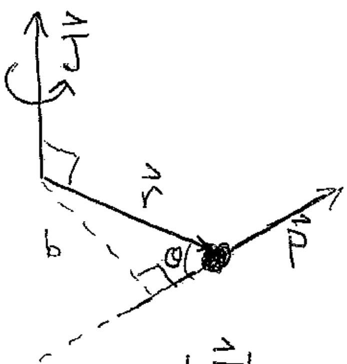
$$\dot{\vec{J}} = m\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = m(\underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} + \underbrace{\vec{r} \times \ddot{\vec{r}}}_{\vec{r} \times \vec{a}})$$

$$= \vec{r} \times \vec{F} \equiv \vec{G} \quad \text{the torque}$$

(or "moment of $\vec{F}$")

i.e. $\boxed{\dot{\vec{J}} = \vec{G}}$

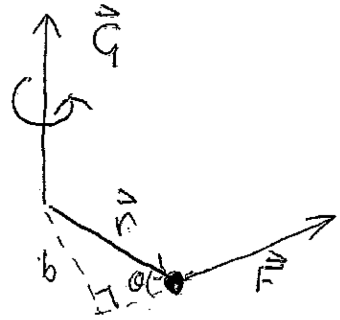
Ang. Mom.



b = perp dist. to origin of $\vec{p}$

$|\vec{J}| = rp\sin\theta$

Torque



$|\vec{G}| = rf\sin\theta$

Central forces: conservation of Ang. Mom.

An external force is CENTRAL if it is always directed towards or away from a fixed point.

If we choose this point to be the origin then $\vec{F}$ is $\parallel$ to $\vec{r}$. Then

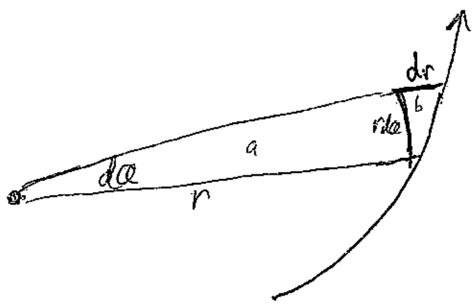
$$\vec{G} = \vec{r} \times \vec{F} = 0$$

$$\Rightarrow \quad \vec{J} = \text{Const.} \qquad \text{Conservation of A.M.}$$

Because $\vec{J}$ is a vector:

1). Direction of $\vec{J}$ is constant. $\vec{J}$ is $\perp$ to plane of $\vec{r}$ and $\vec{v}$
$\Rightarrow$ if $\vec{J}$ is fixed then $\vec{r}$ and $\vec{v}$ always lie in fixed plane
i.e. motion is confined to initial plane.

2) Magnitude of $\vec{J}$ is constant. In a short time interval, the change in the polar coords (r, θ) are dr and $d\theta$.
Distance travelled in radial direction : dr
" " " transverse " : $r\,d\theta$



Area of triangle a
$= \pi r^2 \times \frac{d\theta}{2\pi}$
$= \frac{1}{2} r^2 d\theta$

Area of triangle b
$\simeq r\,d\theta\,dr \approx 0$
(2 small quantities)

$v_r = \dot{r} \qquad V_\theta = r\dot{\theta}$

$$\Rightarrow \quad |\vec{J}| = m r v_\theta = m r^2 \dot{\theta}$$

Geometrical Interpretation of $J = \text{const}$:

. Radius vector sweeps out area $dA = \frac{1}{2} r^2 d\theta$ when angle changes by $d\theta$.

So $\quad \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m} = \text{Const} \qquad$ rate of sweeping out area is const.

Already familiar as Kepler's 2nd Law, although it is more general i.e. applies to motion under any central force.

Alternative statement : $V_\theta \propto \frac{1}{r}$ (from $J = mrV_\theta$)

Central conservative forces — conserve energy T+U — Chpt 4

2 conservation laws ① Energy $\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{const}$

② Ang. Mom. $m\vec{r}\times\dot{\vec{r}} = \vec{J} = \text{const}$

Motion is confined to a plane and so two-dimensional ⇒ use polar coords:

$$① \quad \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

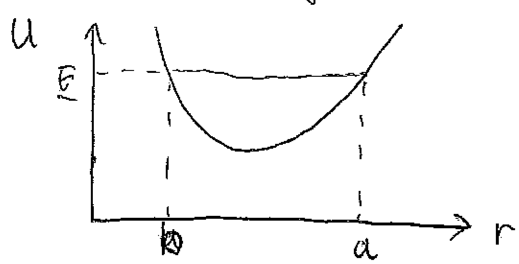
$$② \quad mr^2\dot{\theta} = J$$

Combine : $$\tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$ radial energy eqn.

for fixed J it has form of one-dimensional energy equation with an <u>effective</u> potential energy

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

motion is limited to range of values of r for which $U(r) \leq E$



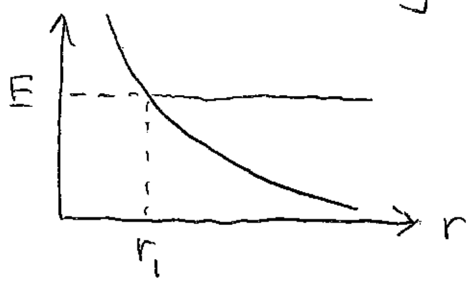
Case of the inverse square law

$$\vec{F} = \frac{k}{r^2}\,\hat{r} \qquad \Rightarrow \quad V(r) = \frac{k}{r}$$

then $$\frac{1}{2} m \dot{r}^2 + \underbrace{\frac{J^2}{2mr^2} + \frac{k}{r}}_{U(r)} = E$$

Repulsive case $k > 0$

$U(r)$ decreases monotonically

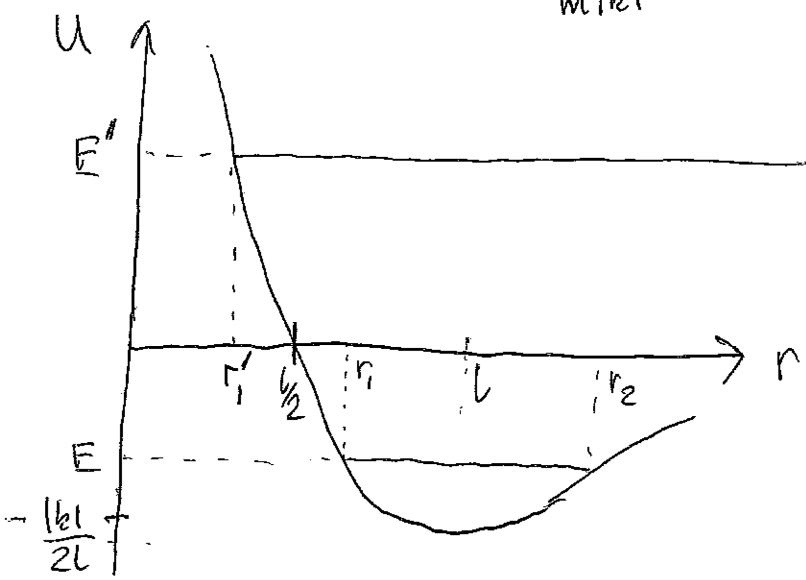


no minima : circular motion impossible.

Orbit is hyperbola (see later)

Attractive case $k < 0$

Define a length $l \equiv \dfrac{J^2}{m|k|}$ $\Rightarrow$ $U(r) = |k|\left(\dfrac{l}{2r^2} - \dfrac{1}{r}\right)$



$U\left(\frac{l}{2}\right) = 0$

Min $U(r)$ is at $r = l$ where $U(l) = -\dfrac{|k|}{2l}$.

Four cases

a) $E = -\frac{|k|}{2l}$ this is min of U $\Rightarrow \dot{r} = 0$

particle moves in circle of radius l

Note $|V| = 2T$ in this case : $V = -\frac{|k|}{l}$

N.B. Virial thm: $\langle V \rangle_{av} + 2\langle T \rangle_{av} = 0$ $T = E - V = \frac{|k|}{2l}$

$2T + V = 0$

Compare SHO : $T + V = E$ $\langle T \rangle = \langle V \rangle$

Heat capacity $\frac{\partial \text{Temp}}{\partial E}$? $E = T + V$ so for gravity

but $V = -2T$

so $E = -T$

Associate Temp with kinetic energy $\langle T \rangle = \frac{3}{2} k_B \text{Temp}$

so $E = -\frac{3}{2} k_B \text{Temp}$

so $\frac{\partial \text{Temp}}{\partial E}$ is $-ve$ $\Rightarrow$ add energy, it cools down!

negative heat capacity

SHO $E = T + V = 2T$ $\frac{\partial T}{\partial E} > 0$

b). $-\frac{|k|}{2L} < E < 0$ (lower line in figure)

radial distance $r_1 < r < r_2$

Orbit is ellipse

c) $E = 0$ min distance is $r_1 = \frac{L}{2}$ but $r_2 = \infty$

Particle can escape to infinity but with $T=0$

orbit is parabola

d) $E > 0$ (upper line, E', in figure)

particle can escape to infinity but with $T > 0$.

orbit is hyperbola.